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COMMENT

Potentials generated by SU(1, 1)

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Abstract. A systematic procedure for deriving a class of potentials with the underlying symmetry group SU(1, 1), starting from the commutation relations for the generators of SU(1, 1), is presented.

Ginnochio (1984) has constructed a class of exactly solvable potentials whose spectral properties are closely related to those of the $\operatorname{sech}^2 x$ potential. Alhassid *et al* (1985) have shown that this class of potentials belongs to the group SU(1, 1). In this comment the constructive procedure of Alhassid *et al* is systematically developed to find potentials with the underlying symmetry group SU(1, 1). It is shown that this approach leads to a larger class of potentials than that discussed by Ginnochio (1984) and Alhassid *et al* (1985).

The SU(1, 1) algebra with the generators $J_\pm$ and J_z is governed by the commutation relations

$$[J_z, J_\pm] = \pm J_\pm \tag{1}$$

and

$$[J_+, J_-] = -2J_z. \tag{2}$$

The Casimir operator is

$$C = J_z^2 - \tfrac{1}{2}(J_+J_- + J_-J_+). \tag{3}$$

Simultaneous eigenstates of the operators C and J_z denoted by $|jm\rangle$ with the eigenvalues $j(j+1)$ and m, respectively, may then be constructed. For discrete representations of SU(1, 1), m can take values $-j+n$ where n is a positive integer. The Hamiltonian H may be taken to be a linear function of the Casimir invariant:

$$H = -\tfrac{1}{4} - C. \tag{4}$$

Then $|jm\rangle$ is also an eigenstate of H with the eigenvalue $-(j+\frac{1}{2})^2$.

To find potentials whose underlying dynamical group is SU(1, 1) one may extend the procedure of Alhassid *et al* (1984) and attempt a representation of the generators of the form

$$J_z = -\mathrm{i}\partial/\partial\varphi \tag{5}$$

$$J_\pm = \exp(\pm \mathrm{i}\varphi)(\pm h(x)\,\mathrm{d}/\mathrm{d}x \pm g(x) + f(x)J_z + c(x)). \tag{6}$$

This representation ensures that (1) is satisfied for any choice of the functions c, f, g and h while (2) is satisfied only if

$$f^2 - h\,\mathrm{d}f/\mathrm{d}x = 1 \tag{7}$$

and

$$h\,\mathrm{d}c/\mathrm{d}x - cf = 0. \tag{8}$$

Explicit solutions of (7) and (8) may be given:

$$f(x) = -\tanh \int_{x_0}^{x} \mathrm{d}y/h(y) \tag{9}$$

$$c(x) = A\,\mathrm{sech} \int_{x_0}^{x} \mathrm{d}y/h(y) \tag{10}$$

in which A and x_0 are arbitrary constants. The Hamiltonian is then given by

$$H = -\tfrac{1}{4} + (f^2 - 1)J_z^2 - h^2\,\mathrm{d}^2/\mathrm{d}x^2 - (h\,\mathrm{d}h/\mathrm{d}x + 2gh - fh)\,\mathrm{d}/\mathrm{d}x$$
$$+ (fg - g^2 - h\,\mathrm{d}g/\mathrm{d}x) + (2cfJ_z + c^2). \tag{11}$$

The requirement that the Hamiltonian be free of terms linear in the momentum operator can be met if

$$g(x) = \tfrac{1}{2}(f - \mathrm{d}h/\mathrm{d}x). \tag{12}$$

It is now clear that a given choice of $h(x)$ leads to a unique determination of $f(x)$, $g(x)$ and $c(x)$ apart from the arbitrary constant A. The eigenvalue equation for H leads to the differential equation

$$\left[\frac{-\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{h^2(x)}\left((m^2 - \tfrac{1}{4} - A^2)\,\mathrm{sech}^2 \int^x \mathrm{d}y/h(y) + 2Am\,\mathrm{sech} \int^x \frac{\mathrm{d}y}{h(y)} \tanh \int^x \frac{\mathrm{d}y}{h(y)}\right)\right.$$
$$\left. + \frac{1}{2}\left(h^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x} h^{-1/2}\frac{\mathrm{d}h}{\mathrm{d}x}\right) + (j + \tfrac{1}{2})^2/h^2(x)\right]\Psi = 0. \tag{13}$$

The next step is to choose $h(x)$ such that (13) reduces to a Schrödinger equation with kinetic and potential terms.

(i) First consider the choice $h(x) = \pm 1$, $x_0 = 0$ which leads to the solutions

$$f = \mp\tanh x \qquad g = \mp\tfrac{1}{2}\tanh x \qquad \text{and} \qquad c = A\,\mathrm{sech}\,x. \tag{14}$$

Equation (13) then becomes the Schrödinger equation for the potential

$$V_m(x, A) = -(m^2 - \tfrac{1}{4} - A^2)\,\mathrm{sech}^2 x - 2Am\,\mathrm{sech}\,x \tanh x \tag{15}$$

with the eigenvalues

$$E_n = -(n + \tfrac{1}{2} - m)^2 \qquad n = 0, 1, \ldots N \leqslant m - \tfrac{1}{2}. \tag{16}$$

For a fixed value of m and varying values of A the family of potentials defined by (15) have identical spectra given by (16). For $A = 0$ (Alhassid *et al* 1984a) $V(x)$ is a $\mathrm{sech}^2 x$ potential. For $A \neq 0$ the potential is no longer a symmetric function of x. For non-zero A the potential is attractive in some region of space $|x| \leqslant \infty$ but is repulsive in some other region of space. (15) also shows that

$$V_{\nu\mu}(x) = -\nu(\nu + 1)\,\mathrm{sech}^2 x + \mu\,\mathrm{sech}\,x \tanh x \tag{17}$$

for fixed values of ν and μ corresponds to

$$m^2=\tfrac{1}{2}\{(\nu+\tfrac{1}{2})^2+[(\nu+\tfrac{1}{2})^4+\mu^2]^{1/2}\}. \tag{18}$$

The spectrum of (17) is then given by (16) with m determined by (18).

(ii) Next consider the choice $h(x)=1$, $x_0=-\infty$ with the corresponding solutions

$$f(x)=-1 \qquad g(x)=-\tfrac{1}{2} \qquad c(x)=\tilde{A}\,e^{-x}. \tag{19}$$

Equation (13) then shows that the Morse potential (Alhassid and Wu 1984) defined by

$$\tilde{V}_m(x)=m^2\{\exp[-2(x-d)]-2\exp[-(x-d)]\} \tag{20}$$

where

$$d=\ln(\tilde{A}/m) \tag{21}$$

has the same spectrum (16) as the family of potentials $V_m(x, A)$ (15) for the same value of m and any value of A.

(iii) A more general solution can be found by noting that if $h(x)$ satisfies the non-linear equation

$$\frac{1}{h^2(x)}=\alpha^2-\frac{1}{h^2(x)}\left(\beta^2\operatorname{sech}^2\int^x\frac{dy}{h(y)}+\gamma\operatorname{sech}\int^x\frac{dy}{h(y)}\tanh\int^x\frac{dy}{h(y)}\right) \tag{22}$$

then (13) becomes the eigenvalue equation of the standard form with kinetic and potential terms. When $\gamma=0$ the explicit solution of (22) is given by

$$\alpha x=\beta z+\tanh^{-1}[(\tan z)/\beta)] \tag{23}$$

where

$$z=\cos^{-1}(h/h_0) \qquad \text{and} \qquad h_0=[(\beta^2+1)/\alpha^2)]^{1/2}. \tag{23a}$$

In terms of a new variable y defined by

$$\alpha x/\beta=\tan^{-1}\beta y+\beta^{-1}\tanh^{-1}y \tag{24}$$

it is easy to show that

$$h=\frac{h_0}{(1+\beta^2y^2)^{1/2}} \qquad f=\frac{-y(1+\beta^2)^{1/2}}{(1+\beta^2y^2)}$$

$$g=-\frac{1}{2\alpha}\left(\frac{1+\beta^2}{1+\beta^2y^2}\right)^{1/2}\left(\alpha y-\alpha\beta^2y\frac{(1-y^2)}{(1+\beta^2)}\right) \qquad \text{and} \qquad c=A\left(\frac{1-y^2}{1+\beta^2y^2}\right)^{1/2}. \tag{25}$$

Correspondence with Alhassid *et al* (1985) may be established by letting

$$\beta^2=\lambda^2-1 \qquad \text{and} \qquad \alpha=\lambda^2. \tag{26}$$

Equation (13) then leads to the Schrödinger equation

$$\{-d^2/dx^2-[(m^2-\tfrac{1}{4}-A^2)+(\lambda^2-1)(j+\tfrac{1}{2})^2]\lambda^2(1-y^2)+2Am\lambda^3y(1-y^2)^{1/2}$$
$$-\tfrac{1}{4}(\lambda^2-1)[5(1-\lambda^2)y^4+(7-\lambda^2)y^2+2]\}\psi=-\lambda^4(j+\tfrac{1}{2})^2\psi. \tag{27}$$

The symmetric potentials discussed by Alhassid *et al* (1985) correspond to the choice $A=0$. For this choice of A let

$$\nu(\nu+1)=(m^2-\tfrac{1}{4})+(\lambda^2-1)(j+\tfrac{1}{2})^2. \tag{28}$$

Since $j(n) = n - m$, fixed values of ν and λ correspond to

$$j(n)+\tfrac{1}{2}=\lambda^{-2}\{(n+\tfrac{1}{2})-[\lambda^2(\nu+\tfrac{1}{2})^2+(1-\lambda^2)(n+\tfrac{1}{2})^2]^{1/2}\}. \tag{29}$$

The spectrum of

$$V=-\nu(\nu+1)\lambda^2(1-y^2)-\tfrac{1}{4}(\lambda^2-1)[5(1-\lambda^2)y^4+(7-\lambda^2)y^2+2] \tag{30}$$

is therefore given by

$$E_n=-(j(n)+\tfrac{1}{2})^2 \tag{31}$$

with $j(n)$ determined by (29).

For $A \neq 0$ the potential in (27) is no longer a symmetric function of x. Let

$$(\delta+\tfrac{1}{2})^2=m^2-A^2+(\lambda^2-1)(j+\tfrac{1}{2})^2 \qquad \text{and} \qquad 2Am=\mu. \tag{32}$$

For fixed values of λ, μ and δ, $j(n)$ is then determined by the quartic equation

$$\lambda^2(j-n)^4+2(j-n)^3(\lambda^2-1)(n+\tfrac{1}{2})$$
$$+(j-n)^2[(\lambda^2-1)(n+\tfrac{1}{2})^2-(\delta+\tfrac{1}{2})^2]-\tfrac{1}{4}\mu^2=0. \tag{33}$$

The eigenvalues of the potential

$$V(x)=-\delta(\delta+1)\lambda^2(1-y^2)+\mu\lambda^3y(1-y^2)^{1/2}$$
$$-\tfrac{1}{4}(\lambda^2-1)[5(1-\lambda^2)y^4+(7-\lambda^2)y^2+2] \tag{34}$$

where

$$\lambda^2x=\tanh^{-1}y+(\lambda^2-1)^{1/2}\tanh^{-1}[y(1-\lambda^2)^{1/2}] \tag{35}$$

are therefore given by (31) with $j(n)$ determined by (33).

Solutions of (22) for $\gamma \neq 0$ will enable the generation of an even larger class of potentials than that discussed above. However we have not been able to solve (22) for non-zero values of γ in closed analytic form.

In this comment it has been shown that a coordinate space realisation of the generators of SU(1, 1) may be used to construct a class of potentials whose spectra are determined by simple algebraic equations. The class of potentials discussed by Ginnochio (1984) and Alhassid *et al* (1985) is a subset of the more general class of potentials discussed in this comment. A similar analysis may be carried out for the group SU(2).

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